A DUALITY PRINCIPLE FOR GROUPS

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ABSTRACT. The duality principle for Gabor frames states that a Gabor sequence obtained by a time-frequency lattice is a frame for $L^2(\mathbb{R}^d)$ if and only if the associated adjoint Gabor sequence is a Riesz sequence. We prove that this duality principle extends to any dual pairs of projective unitary representations of countable groups. We examine the existence problem of dual pairs and establish some connection with classification problems for II_1 factors. While in general such a pair may not exist for some groups, we show that such a dual pair always exists for every subrepresentation of the left regular unitary representation when G is an abelian infinite countable group or an amenable ICC group. For free groups with finitely many generators, the existence problem of such a dual pair is equivalent to the well known problem about the classification of free group von Neumann algebras.

1. Introduction

Motivated by the duality principle for Gabor representations in time-frequency analysis we establish a general duality theory for frame representations of infinite countable groups, and build its connection with the classification problem [2] of II_1 factors. We start by recalling some notations and definitions about frames.

A *frame* [6] for a Hilbert space H is a sequence $\{x_n\}$ in H with the property that there exist positive constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{g\in G} |\langle x\,,\, x_n\rangle|^2 \leq B\|x\|^2 \tag{1.1}$$

holds for every $x \in H$. *A tight frame* refers to the case when $A = B$, and a *Parseval frame* refers to the case when $A = B = 1$. In the case that (1.1) hold only for all $x \in \overline{span}\{x_n\}$, then we say that $\{x_n\}$ is a *frame sequence*, i.e., it is a frame for its closed linear span. If we only require the right-hand side of the inequality (1.1), then $\{x_n\}$ is called a *Bessel sequence.*

One of the well studied classes of frames is the class of Gabor (or Weyl-Heisenberg) frames: Let $\mathcal{K} = A\mathbb{Z}^d$ and $\mathcal{L} = B\mathbb{Z}^d$ be two full-rank lattices in $\mathbb{R}^d$, and let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \mathcal{L} \times \mathcal{K}$. Then the *Gabor (or Weyl-Heisenberg) family* is the following collection of functions in $L^2(\mathbb{R}^d)$:

$$\mathbf{G}(g,\Lambda) = \mathbf{G}(g,\mathcal{L},\mathcal{K}) := \{e^{2\pi i<\ell,x>} g(x-\kappa) \,\Big|\, \ell \in \mathcal{L}, \kappa \in \mathcal{K}\}.$$

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For convenience, we write $g_\lambda = g_{\kappa,\ell} = e^{2\pi i<\ell,x>}g(x-\kappa)$, where $\lambda = (\kappa, \ell)$. If E_ℓ and T_κ are the modulation and translation unitary operators defined by

$$E_\ell f(x) = e^{2\pi i<\ell,x>}f(x)$$

and

$$T_\kappa f(x) = f(x-\kappa)$$

for all $f \in L^2(\mathbb{R}^d)$. Then we have $g_{\kappa,\ell} = E_\ell T_\kappa g$. The well-known Ron-Shen duality principle states that a Gabor sequence $\mathbf{G}(g,\Lambda)$ is a frame (respectively, Parseval frame) for $L^2(\mathbb{R}^d)$ if and only if the adjoint Gabor sequence $\mathbf{G}(g,\Lambda^o)$ is a Riesz sequence (respectively, orthonormal sequence), where $\Lambda^o = (B^t)^{-1}\mathbb{Z}^d \times (A^t)^{-1}\mathbb{Z}^d$ is the adjoint lattice of Λ.

Gabor frames can be viewed as frames obtained by projective unitary representations of the abelian group $\mathbb{Z}^d \times \mathbb{Z}^d$. Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ with A and B being $d \times d$ invertible real matrices. The *Gabor representation* π_Λ defined by $(m,n) \to E_{Am}T_{Bn}$ is not necessarily a unitary representation of the group $\mathbb{Z}^d \times \mathbb{Z}^d$. But it is a projective unitary representation of $\mathbb{Z}^d \times \mathbb{Z}^d$. Recall (cf. [23]) that a *projective unitary representation* π for a countable group G is a mapping $g \to \pi(g)$ from G into the group $U(H)$ of all the unitary operators on a separable Hilbert space H such that $\pi(g)\pi(h) = \mu(g,h)\pi(gh)$ for all $g, h \in G$, where $\mu(g,h)$ is a scalar-valued function on $G \times G$ taking values in the circle group $\mathbb{T}$. This function $\mu(g,h)$ is then called a *multiplier or* 2*-cocycle* of π. In this case we also say that π is a μ-projective unitary representation. It is clear from the definition that we have

(i) $\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mu(g_1g_2, g_3)\mu(g_1, g_2)$ for all $g_1, g_2, g_3 \in G$,

(ii) $\mu(g,e) = \mu(e,g) = 1$ for all $g \in G$, where e denotes the group unit of G.

Any function $\mu : G \times G \to \mathbb{T}$ satisfying (i) – (ii) above will be called a *multiplier* for G. It follows from (i) and (ii) that we also have

(iii) $\mu(g, g^{-1}) = \mu(g^{-1}, g)$ holds for all $g \in G$.

Examples of projective unitary representations include unitary group representations (i.e., $\mu \equiv 1$) and the Gabor representations in time-frequency analysis.

Similar to the group unitary representation case, the left and right regular projective representations with a prescribed multiplier μ for G can be defined by

$$\lambda_g \chi_h = \mu(g,h)\chi_{gh}, \quad h \in G,$$

and

$$\rho_g \chi_h = \mu(h, g^{-1})\chi_{hg^{-1}}, \quad h \in G,$$

where $\{\chi_g : g \in G\}$ is the standard orthonormal basis for $\ell^2(G)$. Clearly, λ_g and r_g are unitary operators on $\ell^2(G)$. Moreover, λ is a μ-projective unitary representation of G with multiplier $\mu(g,h)$ and ρ is a projective unitary representation of G with multiplier $\overline{\mu(g,h)}$. The representations λ and ρ are called the *left regular μ-projective representation* and the *right regular μ-projective representation*, respectively, of G. Let $\mathcal{L}$ and $\mathcal{R}$ be the von Neumann algebras generated by λ and ρ, respectively. It is known (cf. [8]), similarly to the case for regular group representations, that both $\mathcal{R}$ and $\mathcal{L}$ are finite von Neumann algebras, and that $\mathcal{R}$ is the commutant of $\mathcal{L}$. Moreover, if for each $e \neq u \in G$, either $\{vuv^{-1} : v \in G\}$ or $\{\mu(vuv^{-1}, v)\overline{\mu(v,u)} : v \in G\}$ is an infinite set, then both $\mathcal{L}$ and $\mathcal{R}$ are factor von Neumann algebras.

Notations. In this paper for a subset M of a Hilbert space H and a subset $\mathcal{A}$ of $B(H)$ of all the bounded linear operators on H, we will use $[M]$ to denote the closed linear span of M, and $\mathcal{A}'$ to denote the commutant $\{T \in B(H) : TA = AY, \forall A \in \mathcal{A}\}$ of $\mathcal{A}$. So we have $\mathcal{L} = \lambda(G)'' = \rho(G)'$ and $\mathcal{R} = \rho(G)'' = \lambda(G)'$. We also use $\mathcal{M} \simeq \mathcal{N}$ to denote two $*$-isomorphic von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$.

Given a projective unitary representation π of a countable group G on a Hilbert space H, a vector $\xi \in H$ is called a *complete frame vector (resp. complete tight frame vector, complete Parseval frame vector)* for π if $\{\pi(g)\xi\}_{g\in G}$ (here we view this as a sequence indexed by G) is a frame (resp. tight frame, Parseval frame) for the whole Hilbert space H, and is just called a *frame vector (resp. tight frame vector, Parseval frame vector)* for π if $\{\pi(g)\xi\}_{g\in G}$ is a *frame sequence (resp. tight frame sequence, Parseval frame sequence)*. A *Bessel vector* for π is a vector $\xi \in H$ such that $\{\pi(g)\xi\}_{g\in G}$ is Bessel. We will use $\mathcal{B}_\pi$ to denote the set of all the Bessel vectors of π.

For $x \in H$, let Θ_x be the analysis operator for $\{\pi(g)x\}_{g\in G}$ (see section 2). It is useful to note that if ξ and η are Bessel vectors for π, then $\Theta_\eta^*\Theta_\xi$ commutes with $\pi(G)$. Thus, if ξ is a complete frame vector for π, then $\eta := S_\xi^{-1/2}\xi$ is a complete Parseval frame vector for π, where $S_\xi = \Theta_\xi^*\Theta_\xi$ and is called the *frame operator* for ξ (or *Bessel operator* if ξ is a Bessel vector). Hence, a projective unitary representation has a complete frame vector if and only if it has a complete Parseval frame vector. In this paper the terminology *frame representation* refers to a projective unitary representation that admits a complete frame vector.

Proposition 1.1. [8, 21] *Let π be a projective unitary representation π of a countable group G on a Hilbert space H. Then π is frame representation if and only if π is unitarily equivalent to a subrepresentation of the left regular projective unitary representation of G. Consequently, if π is frame representation, then both $\pi(G)'$ and $\pi(G)''$ are finite von Neumann algebras.*

The duality principle for Gabor frames was independent and essentially simultaneous discovered by Daubechies, H. Landau, and Z. Landau [3], Janssen [15], and Ron and Shen [22], and the techniques used in these three articles to prove the duality principle are completely different. We refer to [14] for more details about this principle and its important applications. For Gabor representations, let Λ^o be the adjoint lattice of a lattice Λ. The well-known density theorem (c.f. [13]) implies that one of two projective unitary representations π_Λ and π_{Λ^o} for the group $G = \mathbb{Z}^d \times \mathbb{Z}^d$ must be a frame representation and the other admits a Riesz vector. So we can always assume that π_Λ is a frame representation of $\mathbb{Z}^d \times \mathbb{Z}^d$ and hence $\pi_{\Lambda^{(o)}}$ admits a Riesz vector. Moreover, we also have $\pi_\Lambda(G)' = \pi_{\Lambda^{(o)}}(G)''$, and both representations share the same Bessel vectors. Rephrasing the duality principle in terms of Gabor representations, it states that $\{\pi_\Lambda(m,n)g\}_{m,n\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ if and only if $\{\pi_{\Lambda^{(o)}}(m,n)g\}_{m,n\in\mathbb{Z}^d}$ is a Riesz sequence. Our first main result reveals that this duality principle is not accidental and in fact it is a general principle for any commuting pairs of projective unitary representations.

Definition 1.1. Two projective unitary representations π and σ of a countable group G on the same Hilbert space H are called a *commuting pair* if $\pi(G)' = \sigma(G)''$.

Theorem 1.2. *Let π be a frame representation and (π, σ) be a commuting pair of projective unitary representations of G on H such that π has a complete frame vector which is also a Bessel vector for σ. Then*

(i) $\{\pi(g)\xi\}_{g\in G}$ is a frame sequence if and only if $\{\sigma(g)\xi\}_{g\in G}$ is a frame sequence,

(ii) if, in addition, assuming that σ admits a Riesz sequence, then $\{\pi(g)\xi\}_{g\in G}$ is a frame (respectively, a Parseval frame) for H if and only if $\{\sigma(g)\xi\}_{g\in G}$ is a Riesz sequence (respectively, an orthonormal sequence.

For a frame representation π, we will call (π, σ) a *dual pair* if (π, σ) is a commuting pair such that π has a complete frame vector which is also a Bessel vector for σ, and σ admits a Riesz sequence. We remark that this duality property is not symmetric for π and σ since π is assumed to be a frame representation and σ in general is not. Theorem 1.2 naturally leads to the following existence problem:

Problem 1. *Let G be a infinite countable group and μ be a multiplier for G. Does every μ-projective frame representation π of G admit a dual pair (π, σ)?*

While we maybe able to answer this problem for some special classes of groups, this is in general open due to its connections (See Theorem 1.4) with the classification problem of II_1 factors which is one of the big problems in von Neumann algebra theory. It has been a longstanding unsolved problem to decide whether the factors obtained from the free groups with n and m generators respectively are isomorphic if n is not equal to m with both $n, m > 1$. This problem was one of the inspirations for Voiculescu's theory of free probability. Recall that the fundamental group $F(\mathcal{M})$ of a type II_1 factor $\mathcal{M}$ is an invariant that was considered by Murray and von Neumann in connection with their notion of continuous dimension in [17], where they proved that that $F(\mathcal{M}) = \mathbb{R}_+^*$ when $\mathcal{M}$ is isomorphic to a hyperfinite type II_1 factor, and more generally when it splits off such a factor. For free groups $\mathcal{F}_n$ of n-generators, by using Voiculescus free probability theory [24], Radulescu [20] showed that the fundamental group $F(\mathcal{M}) = \mathbb{R}_+^*$ for $\mathcal{M} = \lambda(\mathcal{F}_\infty)'$. But the problem of calculating $F(\mathcal{M})$ for $M = \lambda(\mathcal{F}_n)'$ with $2 \leq n < \infty$ remains a central open problem in the classification of II_1 factors, and it can be rephrased as:

Problem 2. *Let $\mathcal{F}_n$ ($n > 1$) be the free group of n-generators and $P \in \lambda(\mathcal{F}_n)'$ is a nontrivial projection. Is $\lambda(\mathcal{F}_n)'$ $*$-isomorphic to $P\lambda(\mathcal{F}_n)'P$?*

It is proved in [20] that either all the von Neumann algebras $P\lambda(\mathcal{F}_n)'P$ ($0 \neq P \in \lambda(\mathcal{F}_n)'$) are $*$-isomorphic, or no two of them are $*$-isomorphic. Our second main result established the equivalence of these two problems for free groups.

Theorem 1.3. *Let $\pi = \lambda_P$ be a subrepresentation of the left regular representation of an ICC (infinite conjugate class) group G and $P \in \lambda(G)'$ be a projection. Then the following are equivalent:*
(i) $\lambda(G)'$ and $P\lambda(G)'P$ are isomorphic von Neumann algebras;
(ii) there exists a group representation σ such that (π, σ) form a dual pair.

The above theorem implies that the answer to Problem 1 is negative in general, but is affirmative for amenable ICC groups.

Theorem 1.4. *Let G be a countable group and λ be its left regular unitary representation (i.e. $\mu \equiv 1$). Then we have*

(i) If G is either an abelian group or an amenable ICC group, then for every projection $0 \neq P \in \lambda(G)'$, there exists a unitary representation σ of G such that $(\lambda|_P, \sigma)$ is a dual pair.

(iii) There exist ICC groups (e.g., $G = \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$), such that none of the nontrivial subrepresentations $\lambda|_P$ admits a dual pair.

We will give the proof of Theorem 1.2 in section 2 and the proof requires some resent work by the present authors including the results on parameterizations and dilations of frame vectors [9, 10, 11], and some result results on the "duality properties" for π-orthogonal and π-weakly equivalent vectors [12]. The proofs for Theorem 1.3 and Theorem 1.4 will provided in section 3, and additionally we will also discuss some concrete examples including the subspace duality principle for Gabor representations.

2. The Duality Principle

We need a series of preparations in order to prove Theorem 1.2.

For any projective representation π of a countable group G on a Hilbert space H and $x \in H$, the *analysis operator* Θ_x for x from $\mathcal{D}(\Theta_x)(\subseteq H)$ to $\ell^2(G)$ is defined by

$$\Theta_x(y) = \sum_{g\in G} \langle y, \pi(g)x\rangle \chi_g,$$

where $\mathcal{D}(\Theta_x) = \{y \in H : \sum_{g\in G} |\langle y, \pi(g)x\rangle|^2 < \infty\}$ is the domain space of Θ_x. Clearly, $\mathcal{B}_\pi \subseteq \mathcal{D}(\Theta_x)$ holds for every $x \in H$. In the case that $\mathcal{B}_\pi$ is dense in H, we have that Θ_x is a densely defined and closable linear operator from $\mathcal{B}_\pi$ to $\ell^2(G)$ (cf. [7]). Moreover, $x \in \mathcal{B}_\pi$ if and only if Θ_x is a bounded linear operator on H, which in turn is equivalent to the condition that $\mathcal{D}(\Theta_x) = H$.

Lemma 2.1. [7] *Let π be a projective representation of a countable group G on a Hilbert space H such that $\mathcal{B}_\pi$ is dense in H. Then for any $x \in H$, there exists $\xi \in \mathcal{B}_\pi$ such that*

(i) *$\{\pi(g)\xi : g \in G\}$ is a Parseval frame for $[\pi(G)x]$;*

(ii) $\Theta_\xi(H) = [\Theta_x(\mathcal{B}_\pi)]$.

Lemma 2.2. *Assume that π is a projective representation of a countable group G on a Hilbert space H such that π admits a Riesz sequence and $\mathcal{B}_\pi$ is dense in H. If $[\Theta_\xi(H)] \neq \ell^2(G)$, then there exists $0 \neq x \in H$ such that $[\Theta_x(H)] \perp [\Theta_\xi(H)]$.*

Proof. Assume that $\{\pi(g)\eta\}_{g\in G}$ is a Riesz sequence. Then we have that $\Theta_\eta(H) = \ell^2(G)$ and Θ_η is invertible when restricted to $[\pi(G)\eta]$. Let P be the orthogonal projection from $\ell^2(G)$ onto $[\Theta_\xi(H)]$. Then $P \in \lambda(G)'$ and $P \neq I$. Let $x = \theta_\eta^{-1}P^\perp\chi_e$. Then $x \neq 0$ and $[\Theta_x(H)] \perp [\Theta_\xi(H)]$. □

Lemma 2.3. [7, 10] *Assume that π is a projective representation of a countable group G on a Hilbert space H such that π admits a complete frame vector ξ. If $\{\pi(g)\eta\}_{g\in G}$ is a frame sequence, then there exists a vector $h \in H$ such that η and h are π-orthogonal and $\{\pi(g)(\eta + h)\}_{g\in G}$ is a frame for H.*

Two other concepts are needed.

Definition 2.1. Suppose π is a projective unitary representation of a countable group G on a separable Hilbert space H such that that the set $\mathcal{B}_\pi$ of Bessel vectors for π is dense in H. We will say that two vectors x and y in H are π-*orthogonal* if the ranges of Θ_x and Θ_y are orthogonal, and that they are π-*weakly equivalent* if the closures of the ranges of Θ_x and Θ_y are the same.

The following result obtained in [12] characterizes the π-orthogonality and π-weakly equivalence in terms of the commutant of $\pi(G)$.

Lemma 2.4. *Let π be a projective representation of a countable group G on a Hilbert space H such that $\mathcal{B}_\pi$ is dense in H, and let $x, y \in H$. Then*

(i) *x and y are π-orthogonal if and only if $[\pi(G)'x] \perp [\pi(G)'y]$ (or equivalently, $x \perp \pi(G)'y$);*

(ii) *x and y are π-weakly equivalent if and only if $[\pi(G)'x] = [\pi(G)'y]$,*

We also need the following parameterization result [9, 10, 11].

Lemma 2.5. *Let π be a projective representation of a countable group G on a Hilbert space H and $\{\pi(g)\xi\}_{g\in G}$ is a Parseval frame for H. Then*

(i) $\{\pi(g)\eta\}_{g\in G}$ is a Parseval frame for H if and only if there is a unitary operator $U \in \pi(G)''$ such that $\eta = U\xi$;

(ii) $\{\pi(g)\eta\}_{g\in G}$ is a frame for H if and only if there is an invertible operator $U \in \pi(G)''$ such that $\eta = U\xi$;

(iii) $\{\pi(g)\eta\}_{g\in G}$ is a Bessel sequence if and only if there is an operator $U \in \pi(G)''$ such that $\eta = U\xi$, i.e., $\mathcal{B}_\pi = \pi(G)''\xi$.

As a consequence of Lemma 2.5 we have

Corollary 2.6. *Let π be a frame representation of a countable group G on a Hilbert space H. Then*

(i) $\mathcal{B}_\pi$ is dense in H,

(ii) π has a complete frame vector which is also a Bessel vector for σ if and only if $\mathcal{B}_\pi \subseteq \mathcal{B}_\sigma$.

Proof. (i) follows immediately from Lemma 2.5(iii).

For (ii), assume that $\{\pi(g)\xi\}_{g\in G}$ is a frame for H and $\{\sigma(g)\xi\}_{g\in G}$ is also Bessel. Then for every $\eta \in \mathcal{B}_\pi$, we have by Lemma 2.5 (iii) there is $A \in \pi(G)''$ such that $\eta = A\xi$. Thus $\{\sigma(g)\eta\}_{g\in G} = A\{\sigma(g)\xi\}_{g\in G}$ is Bessel, and so $\eta \in \mathcal{B}_\sigma$. Therefore we get $\mathcal{B}_\pi \subseteq \mathcal{B}_\sigma$. The other direction is trivial. □

Now we are ready to prove Theorem 1.2. We divide the proof into two propositions.

Proposition 2.7. *Let π be a frame representation and (π, σ) be a commuting pair of projective unitary representations of G on H such that π has a complete frame vector which is also a Bessel vector for σ. Then $\{\pi(g)\xi\}_{g\in G}$ is a frame sequence (respectively, a Parseval frame sequence) if and only if $\{\sigma(g)\xi\}_{g\in G}$ is a frame sequence (respectively, a Parseval frame sequence).*

Proof. "$\Rightarrow$:" Assume that $\{\pi(g)\xi\}_{g\in G}$ is a frame sequence. Since π is a frame representation, by the dilation result (Lemma 2.3), there exists $h \in H$ such that (ξ, h) are π-orthogonal and $\{\pi(g)(\xi + h)\}_{g\in G}$ is a frame for H. If we prove that $\{\sigma(g)(\xi + h)\}_{g\in G}$ is a frame sequence, then $\{\sigma(g)\xi\}_{g\in G}$ is a frame sequence. In fact, using the π-orthogonality of ξ and h and Lemma 2.4, we get that $[\pi(G)'\xi] \perp [\pi(G)'h]$, and hence $[\sigma(G)\xi] \perp [\sigma(G)h]$ since $\sigma(G)'' = \pi(G)'$. Therefore, projecting $\{\sigma(g)(\xi+h)\}_{g\in G}$ onto $[\sigma(G)\xi]$ we get that $\{\sigma(g)\xi\}_{g\in G}$ is a frame sequence as claimed. Thus, without losing the generality, we can assume that $\{\pi(g)\xi\}_{g\in G}$ is a frame for H.

By Corollary 2.6, we have $\xi \in \mathcal{B}_\pi \subseteq \mathcal{B}_\sigma$. From Lemma 2.1 we can choose $\eta \in [\sigma(G)\xi] =: M$ such that ξ and η are σ-weakly equivalent and $\{\sigma(g)\eta\}_{g\in G}$ is a Parseval frame for $[\sigma(G)\xi]$. By the parameterizition theorem (Lemma 2.5) there exists an operator $A \in \sigma(G)''|_M$ such that $\xi = A\eta$. Assume that C is the lower frame bound for $\{\pi(g)\xi\}_{g\in G}$. Then for every $x \in M$ we have

$$\begin{aligned} ||x||^2 &\leq \frac{1}{C}\sum_{g\in G} |\langle x,\ \pi(g)\xi\rangle|^2 = \frac{1}{C}\sum_{g\in G} |\langle x,\ \pi(g)A\eta\rangle|^2 \\ &= \frac{1}{C}\sum_{g\in G} |\langle A^*x,\ \pi(g)\eta\rangle|^2 = \frac{1}{C}||A^*x||^2. \end{aligned}$$

Thus A^* is bounded from below and therefore it is invertible since $\sigma(G)''|_M$ is a finite von Neumann algebra (Proposition 1.1). This implies that A is invertible (on M) and so $\{\sigma(g)\xi\}_{g\in G}(= \{A\pi(g)\eta\}_{g\in G}$ is a frame for M.

"$\Leftarrow$:" Assume that $\{\sigma(g)\xi\}_{g\in G}$ is a frame sequence. Applying Lemma 2.1 again there exists $\eta \in [\pi(G)\xi]$ such that η and ξ are π-weakly equivalent, and $\{\pi(g)\eta\}_{g\in G}$ is a Parseval frame for $[\pi(G)\xi]$. Using the converse statement proved above, we get that $\{\sigma(g)\eta\}_{g\in G}$ is a frame sequence for $M := [\sigma(G)\eta]$. Since ξ are π-weakly equivalent, we have by Lemma 2.4 that $[\pi(G)'\xi] = [\pi(G)'\eta]$ and so $M = [\sigma(G)\eta] = [\sigma(G)\xi]$. Thus $\{\sigma(g)\eta\}_{g\in G}$ is a frame for $[\sigma(G)\xi]$. By the parameterization theorem (Lemma 2.5), there exists an invertible operator operator $A \in \sigma(G)''|_M$ such that $\xi = A\eta$. Extending A to an invertible operator B in $\sigma(G)''$, we have $A\eta = B\eta$, and so

$$\pi(g)\xi = \pi(g)A\eta = \pi(g)B\eta = B\pi(g)\eta.$$

Thus $\{\pi(g)\xi\}_{g\in G}$ is a frame sequence since $\{\pi(g)\eta\}_{g\in G}$ is a frame sequence and B is bounded invertible.

For the Parseval frame sequence case, all the operators A and B involved in the parameterization are unitary operators and the rest of the argument is identical to the frame sequence case. □

Proposition 2.8. *Let π be a frame representation of G on H. Assume that (π, σ) is a commuting pair of projective unitary representations of G on H such that such that π has a complete frame vector which is also a Bessel vector for σ. If σ admits a Riesz sequence, then*

(i) $\{\pi(g)\xi\}_{g\in G}$ is a frame for H if and only if $\{\sigma(g)\xi\}_{g\in G}$ is a Riesz sequence.

(ii) $\{\pi(g)\xi\}_{g\in G}$ is a Parseval frame for H if and only if $\{\sigma(g)\xi\}_{g\in G}$ is an orthonormal sequence.

Proof. (i) "$\Rightarrow$:" Assume that $\{\pi(g)\xi\}_{g\in G}$ is a frame for H. Then from Proposition 2.7 we have that $\{\sigma(g)\xi\}_{g\in G}$ is a frame sequence.

Thus, in order to show that $\{\sigma(g)\xi\}_{g\in G}$ is a Riesz sequence, it suffices to show that $[\Theta_{\sigma,\xi}(H)] = \ell^2(G)$, where $\Theta_{\sigma,\xi}$ is the analysis operator of $\{\sigma(g)\xi\}_{g\in G}$. We prove this by contradiction.

Assume that $[\Theta_{\sigma,\xi}(H)] \neq \ell^2(G)$. Then, by Lemma 2.2, there is a vector $0 \neq x \in H$ such that $\Theta_{\sigma,x}(H) \perp \Theta_{\sigma,\xi}(H)$. Since $\mathcal{B}_\sigma$ is dense in H (recall that $\mathcal{B}_\pi$ is dense in H since π is a frame representation), we get by Lemma 2.4 that $[\sigma(G)'x] \perp [\sigma(G)'\xi]$ and so $[\pi(G)x] \perp [\pi(G)\xi]$ since $\sigma(G)' = \pi(G)''$. On the other hand, since $\{\pi(g)\xi\}_{g\in G}$ is a frame for H, we have $[\pi(G)\xi] = H$ and so we have $x = 0$, a contradiction.

"$\Leftarrow$:" Assume that $\{\sigma(g)\xi\}_{g\in G}$ is a Riesz sequence. Then, again by Proposition 2.7 we $\{\pi(g)\xi\}_{g\in G}$ is a frame sequence. So we only need to show that $[\pi(G)\xi] = H$.

Let $\eta \perp [\pi(G)\xi]$. So we have $[\pi(G)\eta] \perp [\pi(G)\xi]$. By Lemma 2.4, we have that $\Theta_{\sigma,\eta}(H) \perp \Theta_{\sigma,\xi}(H)$. But $\Theta_{\sigma,\xi}(H) = \ell^2(G)$ since $\{\sigma(g)\xi\}_{g\in G}$ is a Riesz sequence. This implies that $\eta = 0$, and so $[\pi(G)\xi] = H$, as claimed.

(ii) Replace "frame" by "Parseval frame", and "Riesz" by "orthonormal", the rest is exactly the same as in (i). $\square$

3. The Existence Problem

We will divide the proof of Theorem 1.4 into two cases: The abelian group case and the ICC group case. We deal the abelian group first, and start with an simple example when $G = Z$.

Example 3.1. Consider the unitary representation of $\mathbb{Z}$ defined by $\pi(n) = M_{e^{2\pi int}}$ on the Hilbert space $L^2[0,1/2]$. Then $\sigma(n) = M_{e^{2\pi i2nt}}$ is another unitary representation of $\mathbb{Z}$ on $L^2[0,1/2]$. Note that $\{\sigma(n)1_{[0,1/2]}\}_{n\in\mathbb{Z}}$ is an orthogonal basis for $L^2[0,1/2]$. We have that $\sigma(Z)''$ is maximal abelian and hence $\sigma(Z)'' = \mathcal{M}_\infty = \pi(Z)'$. Moreover a function $f \in L^2[0,1/2]$ is a Bessel vector for π (respectively, σ) if and only if $f \in L^\infty[0,1/2]$. So π and σ share the same Bessel vectors. Therefore (π,σ) is a commuting pair with the property that $\mathcal{B}_\pi = \mathcal{B}_\sigma$, and σ admits a Riesz sequence.

It turns out the this example is generic for abelian countable discrete group.

Proposition 3.1. *Let π be a unitary frame representation of an abelian infinite countable discrete group G on H. Then there exists a group representation σ such that (π,σ) is a dual pair.*

Proof. Let $\hat{G}$ be the dual group of G. Then $\hat{G}$ is a compact space. Let μ be the unique Haar measure of $\hat{G}$. Any frame representation π of G is unitarily equivalent to a representation of the form: $g \to e_g|_E$, where E is a measurable subset of $\hat{G}$ with positive measure, and e_g is defined by $e_g(\chi) =< g,\chi >$ for all $\chi \in \hat{G}$. So without losing the generality, we can assume that $\pi(g) = e_g|_E$.

Let $\nu(F) := \frac{1}{\mu(E)}\mu(F)$ for any measurable subset F of E. Then both μ and ν are Borel probability measures without any atoms. Hence (see [5]) there is a measure preserving bijection ψ from E onto $\hat{G}$. Define a unitary representation σ of G on $L^2(E)$ by

$$\sigma(g)f(\chi) = e_g(\psi(\chi))f(\chi), \quad f \in L^2(E).$$

Then by the same arguments as in Example 3.1 we have that $\{\sigma(g)1_E\}_{g\in G}$ is an orthogonal basis for $L^2(E)$, and (π, σ) satisfies all the requirements of this theorem. $\square$

Proof of Theorem 1.3

"(i) $\Rightarrow$ (ii):" Let $\Phi : \lambda(G)' \to P\lambda(G)'P$ be an isomorphism between the two von Neumann algebras. Note that $tr(A) =< A\chi_e, \chi_e >$ is a normalized normal trace for $\lambda(G)'$. Define τ on $\lambda(G)'$ by

$$\tau(A) = \frac{1}{tr(P)} tr(\Phi(A)), \quad \forall A \in \lambda(G)'.$$

Then τ is also an normalized normal trace for $\lambda(G)'$. Thus $\tau(\cdot) = tr(\cdot)$ since $\lambda(G)'$ is a factor von Neumann algebra. In particular we have that

$$\frac{1}{tr(P)} tr(\Phi(\rho_g)) = \tau(\rho_g) = tr(\rho_g) = \delta_{g,e}.$$

Therefore, if we define $\sigma(g) = \Phi(\rho_g)$, then σ is a unitary representation of G such that $\sigma(G)'' = P\lambda(G)'P = (\lambda(G)P)' == \pi(G)'$ and σ admits an orthogonal sequence $\{\sigma(g)\xi\}_{g\in G}$, where $\xi = P\chi_e$. Moreover, for any $A \in \pi(G)''$ we have that $\sigma(g)A\xi = A\sigma(g)\xi$ and so $A\xi$ is a Bessel vector for σ. By Lemma 2.5 (iii), we know that $\mathcal{B}_\pi = \pi(G)''\xi$. Thus we get $\mathcal{B}_\pi \subseteq \mathcal{B}_\sigma$ and therefore (π, σ) is a dual pair.

"(ii) $\Rightarrow$ (i):" Assume that (π, σ) is a dual pair. Let $\{\sigma(g)\psi\}_{g\in G}$ be a Riesz sequence, $\sigma_1(g) := \sigma(g)|_M$ and $\sigma_2(g) := \sigma(g)|_{M^\perp}$, where $M = [\sigma(g)\psi]$. Then σ is unitarily equivalent to the group representation $\zeta := \sigma_1 \oplus \sigma_2$ acting on the Hilbert space $K := M \oplus M^\perp$. Since σ_1 is unitarily equivalent to the right regular representation of G (because of the Riesz sequence), we have that $\sigma_1(G)'' \simeq \lambda(G)'$. Let q be the orthogonal projection from K onto $M \oplus 0$. Then $q \in \zeta(G)'$. Clearly, $\zeta(G)''q \simeq \sigma_1(G)''$. Since $\zeta(G)''$ is a factor, we also have that $\zeta(G)'' \simeq \zeta(G)''q$, and hence $\sigma(G)'' \simeq \lambda(G)'$, i.e., $\lambda(G)' \simeq P\lambda(G)'P$ since $\sigma(G)'' = P\lambda(G)'P$. $\square$

Remark 3.1. Although we stated the result in Theorem 1.3 for group representations, the proof works for general projective unitary representations when the von Neumann algebra generated by the left regular projective unitary representation of G is a factor.

Proof of Theorem 1.4

(i) The abelian group case is proved in Proposition 3.1. If G is an amenable ICC group G, then the statement follows immediately from Theorem 1.3 and the famous result of A. Connes [2] that when G is an amenable ICC group, then $\lambda(G)'$ is the hyperfinite II_1 factor, and we have that $\lambda(G)'$ and $P\lambda(G)'P$ are isomorphic for any non-zero projection $P \in \lambda(G)'$.

(ii) Recall that the fundamental group of a type II_1 factor $\mathcal{M}$ is the set of numbers $t > 0$ for which the amplification of $\mathcal{M}$ by t is isomorphic to $\mathcal{M}$, $F(\mathcal{M}) = \{t > 0 : \mathcal{M} \simeq \mathcal{M}^t\}$. Let $G = \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$. Then, by [18, 19], the fundamental group of $\lambda(G)'$ is $\{1\}$, which

implies that von Neumann algebras $P\lambda(G)'$ is not $*$-isomorphic to $\lambda(G)'$ for any nontrivial projection $P \in \lambda(G)'$. Thus, by Theorem 1.3, none of the nontrivial subrepresentations $\lambda|_P$ admits a dual pair. $\square$

Example 3.2. Let $G = \mathcal{F}_\infty$. Using Voiculescus free probability theory, Radulescu [20] proved that fundamental group $F(\mathcal{M}) = \mathbb{R}^*_+$ for $\mathcal{M} = \lambda(\mathcal{F}_\infty)'$. Therefore for $\lambda(\mathcal{F}_\infty)' \simeq P\lambda(\mathcal{F}_\infty)'P$ for any nonzero projection $P \in \lambda(\mathcal{F}_\infty)'$, and thus $\lambda|_P$ admits a dual pair for free group $\mathcal{F}_\infty$.

Example 3.3. Let $G = \mathbb{Z}^d \times \mathbb{Z}^d$, and $\pi_\Lambda(m,n) = E_m T_n$ be the Gabor representation of G on $L^2(\mathbb{R}^d)$ associated with the time-frequency lattice $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$. Since G is abelian, we have that the von Neumann algebra $\pi_\Lambda(G)'$ is amenable (cf. [1]). Thus, if $\pi_\Lambda(G)'$ is a factor, then for every π_Λ invariant subspace M of $L^2(\mathbb{R}^d)$, we have by the remark after the proof of Theorem 1.3 that $\pi_\Lambda|_M$ admits a dual pair. Therefore the duality principle in Gabor analysis holds also for subspaces at least for the factor case (e.g., $d = 1$, $A = a$ and $B = b$ with ab irrational). In the case that $A = B = I$, then the Gabor representation π_Λ is a unitary representation of the abelian group $\mathbb{Z}^d \times \mathbb{Z}^d$, and so, from Proposition 3.1, the duality principle holds for subspaces for this case as well, In fact in this case a concrete representation σ can be constructed by using the Zak transform.

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References

[1] E. Bédos and R. Conti, On twisted Fourier analysis and convergence of Fourier series on discrete groups, preprint, 2008.

[2] A. Connes, Classification of injective factors. Cases $II_1, II_\infty, III_\lambda, \lambda \neq 1$, *Ann. of Math.,* 104(1976), 73–115.

[3] I. Daubechies, H. Landau and Z. Landau, Gabor time-frequency lattices and the Wexler-Raz identity, *J. Fourier Anal. Appl.,* 1 (1995), 437–478.

[4] D. Dutkay, D. Han and G. Picioroaga, Frames for ICC groups, *J. Funct. Analysis,* to appear

[5] J. Dixmier, Von Neumann Algebras, With a preface by E. C. Lance. Translated from the second French edition by F. Jellett. North-Holland Mathematical Library, 27. North-Holland Publishing Co., Amsterdam-New York, 1981. xxxviii+437 pp.

[6] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.,* 72(1952), 341–366.

[7] J-P. Gabardo and D. Han, Subspace Weyl-Heisenberg frames, *J. Fourier Anal. Appl.,* 7 (2001), 419–433.

[8] J-P. Gabardo and D. Han, Frame representations for group-like unitary operator systems, *J. Operator Theory,* 49(2003), 223-244.

[9] J-P. Gabardo and D. Han, The uniqueness of the dual of Weyl-Heisenberg subspace frames, *Appl. Comput. Harmon. Anal.,* 17 (2004), 226–240.

[10] D. Han, Frame Representations and Parseval Duals with Applications to Gabor Frames, *Trans. Amer. Math. Soc.,* 360(2008), 3307–3326.

[11] D. Han and D. Larson, Frames, bases and group representations, *Memoirs Amer. Math. Soc.,* 697 (2000).

[12] D. Han and D. Larson, Frame duality properties for projective unitary representations, *Bull. London Math. Soc.,* 40(2008), 685–695.

[13] D. Han and Y. Wang, Lattice tiling and Weyl-Heisenberg frames, *Geometric and Functional Analysis,* 11(2001), 742–758.
[14] C. Heil, History and evolution of the density theorem for Gabor frames, *J. Fourier Anal. Appl.,* 13 (2007), 113–166.
[15] A. Janssen, Duality and biorthogonality forWeyl-Heisenberg frames, *J. Fourier Anal. Appl.,* 1(1995), 403-436.
[16] R. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vol. I and II,* Academic Press, Inc. 1983 and 1985.
[17] F.J. Murray and J. von Neumann, On rings of operators. IV, *Ann. Math.,* 44(1943), 716808.
[18] S. Popa, On the fundamental group of type II_1 factors, *Proc. Natl. Acad. Sci.,* 101 (2004), 723–726.
[19] S. Popa, On a class of type II_1 factors with Betti numbers invariants, *Ann. of Math.,* 163 (2006), 809-899
[20] F. Radulescu, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index, *Invent. Math.,* 115 (1994),347–389.
[21] M. A. Rieffel, Von Neumann algebras associated with pairs of lattices in Lie groups, *Math. Ann.,* 257(1981), 403-413
[22] A. Ron and Z. Shen, Weyl–Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$, *Duke Math. J.,* 89(1997) 237–282.
[23] V. S. Varadarajan, *Geometry of Quantum Theory,* Second Edition, Springer-Verlag, New York–Berlin, 1985.
[24] D.V. Voiculescu, K.J. Dykema, and A. Nica, Free random variables, *CRM Monograph Series, A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.* Amrican Mathematical Society, Providence, RI, 1992, vi+70

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